Theorem 2.17 (Necessary Conditions for Local Optimality). If $f(x)$ is twice continuously differentiable and there exists a point x^* that is a local minimum, then $\nabla f(x^*) = 0$ and $\nabla^2 f(x^*)$ must be positive semidefinite.

Claim 1: $\nabla f(x^*) = 0$.
Overall approach: assume x^* is a local minimizer and $\nabla f(x^*) \neq 0$. Establish contradiction.

Step 1. Write Taylor series around $f(x^*)$ to calculate $f(x^* + t\,p)$

Step 2. Choose $p = -\nabla f(x^*)$ and substitute into Taylor series

Step 3. Choose t sufficiently small so that $\nabla f(\cdot)$ does not change sign in the Taylor series

Step 4. Establish contradiction with definition of local minimizer.

Claim 2: $\nabla^2 f(x^*)$ must be positive semidefinite. Overall approach is same as above.

Step 1: Write Taylor series with $\nabla^2 f(\cdot)$

Step 2: Choose t sufficiently small so that $\nabla^2 f(\cdot)$ does not change signs due to continuity

Step 3: Establish contradiction.

Theorem 2.18 (Sufficient Conditions for Local Optimality). If $f(x)$ is continuously twice differentiable and there exists a point x^* where $\nabla f(x^*) = 0$ and $\nabla^2 f(x^*)$ is positive definite, then x^* is a strict, isolated local minimum.

Claim 1: strict local minimizer

Step 1: Write Taylor series

Step 2: Apply $\nabla f(x^*) = 0$

Step 3: Choose p sufficiently small to evoke continuity of the second derivative

Step 4: Recover definition of strict local minimizer

Claim 2: isolated local minimizer

Step 1: Consider neighborhood $N(x^*)$ where $\nabla^2 f(x)$ is P.D. for all $x \in N(x^*)$

Step 2: Assume local minimizers $x^*, x^+ \in N(x^*)$

Step 3: Write Taylor series (integral form)

Step 4: Evoke that $\nabla f(x^*) = \nabla f(x^+) = 0$.

Step 5: Contradiction with $\nabla^2 f(x)$ is P.D.