Error Propagation Example

CBE 20258 / CBE 40258 / CBE 60258

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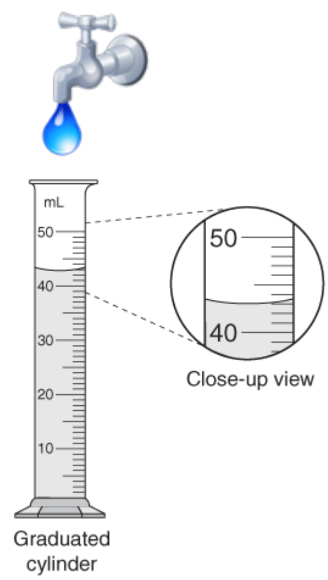
1 Experimental Setup

Imagine you want to measure the flow rate of water out of a faucet, but your flowmeter is broken. Quick on your feet, you devise a simple experiment using a graduate cylinder and wrist watch. The average flow rate is:

$$F = \frac{V_f - V_0}{t_f - t_0} \tag{1}$$

where V_0 is the volume in the cylinder at time t_0. You measure V_f and t_f after some time has elapsed.



2 Error Propagation

Your graduated cylinder only measures to the nearest 0.1 mL and holds a maximum of 50 mL. Your analog wristwatch has a seconds hand. Based on this, you reason that $\sigma_V = \underline{\hspace{1.5em}}$ mL (same for both volume measurements), $\sigma_t = \underline{\hspace{1.5em}}$ s (same for both time measurements), and $\sigma_{V,t} = \underline{\hspace{1.5em}}$ (i.e., the measurement errors are NOT correlated). Although you *could* conduct many experiments to estimate σ_V, σ_t, and $\sigma_{V,t}$, you think it is wise to *approximate* the error level in F after taking only a few measurements. Here are the data for your first trial:

V_0	t_0	V_f	t_f
5.3 mL	1:01:04	49.2 mL	1:01:56

2.a Approach 1

Which error propagation formula to use? You decide to decompose (1) into two subtraction steps and one division step:

$$\Delta t := t_f - t_0, \qquad \Delta V := V_f - V_0, \qquad F = \frac{\Delta V}{\Delta t} \tag{2}$$

where ΔV and Δt are intermediate variables. You then apply the subtraction error propagation formula to calculate $\sigma_{\Delta V}$ and $\sigma_{\Delta t}$.

Next, apply the division error propagation formula to estimate σ_F:

2.b Approach 2

What about the general error propagation formula,

$$\sigma_Z^2 \approx \left|\frac{\partial g}{\partial X}\right|^2 \cdot \sigma_X^2 + \left|\frac{\partial g}{\partial Y}\right|^2 \cdot \sigma_Y^2 + 2 \cdot \frac{\partial g}{\partial X} \cdot \frac{\partial g}{\partial Y} \cdot \sigma_{X,Y} \tag{3}$$

where $Z = g(X, Y)$ and $g(\cdot)$ is a differentiable function? You apply this formula to (1). Start by calculating the partial derivatives $\frac{\partial F}{\partial V_f}$, $\frac{\partial F}{\partial V_0}$, $\frac{\partial F}{\partial t_f}$, $\frac{\partial F}{\partial t_0}$:

Next, apply the general formula, (3), and drop the covariance terms (assumed to be zero).

$$\sigma_F^2 \approx \left|\frac{\partial F}{\partial V_f}\right|^2 \sigma_V^2 + \left|\frac{\partial F}{\partial V_0}\right|^2 \sigma_V^2 + \left|\frac{\partial F}{\partial t_f}\right|^2 \sigma_t^2 + \left|\frac{\partial F}{\partial t_0}\right|^2 \sigma_t^2$$

Finally, substitute and simplify:

3 Calculation

Now that we have confirmed Approach 1 and Approach 2 yield the same error propagation formula, we can perform the calculation. Recall the data for the first trial:

V_0	t_0	V_f	t_f
5.3 mL	1:01:04	49.2 mL	1:01:56

and our assumptions $\sigma_V = 0.1$ mL and σ_t = 1 s.

Calculate F and estimate σ_F:

$$F = \frac{\Delta V}{\Delta t}, \qquad \sigma_F^2 \approx F^2 \left[2 \left(\frac{\sigma_V}{\Delta V} \right)^2 + 2 \left(\frac{\sigma_t}{\Delta t} \right)^2 \right]$$